九十六學年第二學期 PHYS2320 電磁學 第一次期中考(共兩頁)

[Griffiths Ch. 7-8]　2008/04/1,　10:10am–12:00am, 教師：張存續

記得寫上學號，班別及姓名等。請依題號順序每頁答一題。

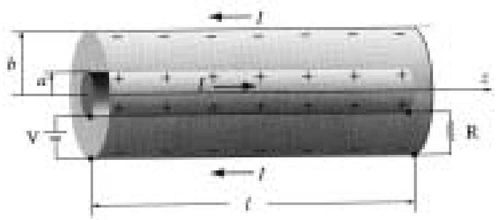
1. (8%, 8%, 4%)

A long coaxial cable of length l consists of an inner conductor (radius a) and an outer conductor (radius b). It is connected to a battery at one end and a resistor at the other, as shown in the figure below. The inner conductor carries a uniform charge per unit length λ, and a steady current I to the right; the outer conductor has the opposite charge and current. [Hint: assume the two conductors are held at a potential difference V.]

(a) Calculate the **E** and **B** fields using Gauss's law and Ampere's law.

(b) Calculate the power (energy per unit time) transported down the cable.

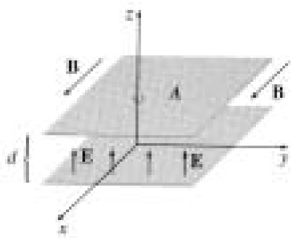
(c) Calculate the energy density u_{em}.



2. (10%, 10%) A charged parallel-plates capacitor (with uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$) is placed in a uniform magnetic field $\mathbf{B} = B_0\hat{\mathbf{x}}$, as shown in the figure.

(a) Show the equation of motion for a particle of charge q and mass m. [Hint: use the velocities as variables and write down the equations of motion in Cartesian coordinate].

(b) Solve the coupled differential equations. If the particle is initially at rest, describe how it moves. [Hint: This is the so-called **E**x**B** drift]

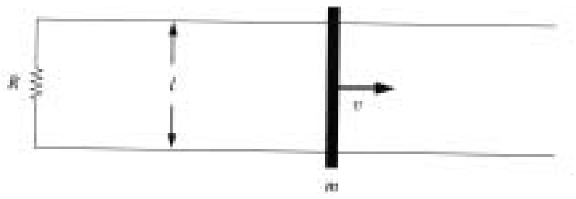


3. (6%, 7%, 7%) A metal bar of mass m slides frictionlessly on two parallel conducting rails a distance l apart (see figure below). A resistor R is connected across the rails and a uniform magnetic field **B**, pointing into the page, fills the entire region.

   (a) If the bar moves to the right at speed v, what is the current in the resistor? In what direction does it flow?

   (b) What is the magnetic force on the bar? [Hint: to speed up or to slow down the movement].

(c) If the bar starts out with speed v_0 at time t=0, and is left to slide, what is its speed at a later time t?

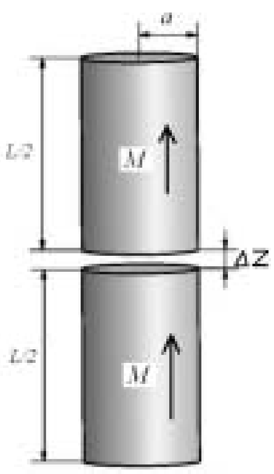


4. (7%, 7%, 6%)
   (a) Write down Maxwell's equations in terms of free charges ρ_f and current $\mathbf{J}_f$. Also for linear media, write the appropriate constitutive relations, giving **D** and **H** in terms of **E** and **B**.
   (b) Write down the four boundary conditions ($E^{\perp}$, $E^{//}$, $B^{\perp}$, and $B^{//}$) for linear media, if there is no free charge and no free current at the interface.
   (c) Write down the equations for conservation of charge, energy, and momentum. Please explain the symbols you use as clear as possible.

5. (6%, 7%, 7%) A cylindrical magnet of length L and radius a carries a uniform magnetization **M** parallel to it axis. It is then cut into two pieces of equal length.
(a) Determine the magnetic field **B** in the gap.
(b) Find the force between these two magnets using the Maxwell stress tensor.
(c) Find the force between them using the concept that $\mathbf{F} = -\nabla U$.

[Hint 1: Maxwell's stress tensor $T_{ij} \equiv \varepsilon_0 (E_i E_j - \frac{1}{2}\delta_{ij}E^2) + \frac{1}{\mu_0}(B_i B_j - \frac{1}{2}\delta_{ij}B^2)$;

Hint 2: attractive or repulsive force?]



1. (a)

Gauss's law: $\int \mathbf{E}\cdot d\mathbf{a} = \frac{Q}{\varepsilon_0},\ E2\pi rl = \frac{\lambda l}{\varepsilon_0} \Rightarrow\ \mathbf{E} = \frac{\lambda}{2\pi r\varepsilon_0}\hat{\mathbf{r}}$

Ampere's law: $\oint \mathbf{B}\cdot d\ell = \mu_0 I,\ B2\pi r = \mu_0 I \Rightarrow\ \mathrm{B} = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\varphi}}$

$$V = \int \mathbf{E}\cdot d\ell = \int_a^b \frac{\lambda}{2\pi r\varepsilon_0}dr = \frac{\lambda}{2\pi\varepsilon_0}\ln(b/a), \Rightarrow \lambda = \frac{2\pi\varepsilon_0 V}{\ln(b/a)}\ \therefore \mathbf{E} = \frac{V}{r\ln(b/a)}\hat{\mathbf{r}}$$

(b)

$$\mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{1}{\mu_0}\frac{V}{r\ln(b/a)}\frac{\mu_0 I}{2\pi r}\hat{\mathbf{r}}\times\hat{\boldsymbol{\varphi}} = \frac{VI}{2\pi r^2\ln(b/a)}\hat{\mathbf{z}}$$

$$P = \int \mathbf{S}\cdot d\mathbf{a} = \int_a^b \mathbf{S}\cdot d\mathbf{a} = \int_a^b \frac{VI}{2\pi r^2\ln(b/a)}2\pi r dr = VI$$

(c)

$$u_{\text{em}} = \frac{1}{2}(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2) = \frac{\varepsilon_0}{2}(\frac{V}{r\ln(b/a)})^2 + \frac{1}{2\mu_0}(\frac{\mu_0 I}{2\pi r})^2$$

2. (a)

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \quad \Rightarrow \begin{cases} m\dot{v}_x = 0 \\ m\dot{v}_y = qv_zB_0 \\ m\dot{v}_z = qE_z - qv_yB_0 \end{cases}$$

(b)

x component: $m\dot{v}_x = 0,\ v_x = const.$

y and z components are coupled: $\begin{cases} m\dot{v}_y = qv_zB_0 \\ m\dot{v}_z = qE_0 - qv_yB_0 \end{cases}$

$$\begin{cases} m\ddot{v}_y = q\dot{v}_zB_0 = \frac{qB_0}{m}(qE_0 - qv_yB_0) \\ m\ddot{v}_z = -q\dot{v}_yB_0 = -\frac{qB_0}{m}qv_zB_0 \end{cases} \Rightarrow \begin{cases} \ddot{v}_y = (\frac{qB_0}{m})^2(\frac{E_0}{B_0} - v_y) \\ \ddot{v}_z = -(\frac{qB_0}{m})^2 v_z \end{cases}$$

Let $(\frac{qB_0}{m}) \equiv \omega_c$ $\quad \begin{cases} \ddot{v}_y = -\omega_c^2(v_y - \frac{E_0}{B_0}) \\ \ddot{v}_z = -\omega_c^2 v_z \end{cases} \Rightarrow \begin{cases} v_y = A\cos(\omega_c t + \phi_1) + \frac{E_0}{B_0} \\ v_z = B\cos(\omega_c t + \phi_2) \end{cases}$

Initial condition ($t=0$): $\begin{cases} (v_x, v_y, v_z) = (0,0,0) \\ (\dot{v}_x, \dot{v}_y, \dot{v}_z) = (0,0,\frac{qE_0}{m}) \end{cases}$ $\begin{cases} v_y = A\cos(\omega_c t+\phi_1)+\frac{E_0}{B_0} \\ v_z = B\cos(\omega_c t+\phi_2) \end{cases}$

x component: $v_x = 0$.

y component: $\begin{cases} v_y = A\cos(\phi_1)+\frac{E_0}{B_0}=0 \\ \dot{v}_y = -A\omega_c \sin(\phi_1)=0 \end{cases} \Rightarrow v_y = -\frac{E_0}{B_0}\cos(\omega_c t)+\frac{E_0}{B_0}$

z component: $\begin{cases} v_z = B\cos(\phi_2)=0 \\ \dot{v}_z = -B\omega_c \sin(\phi_2)=\frac{qE_0}{m} \end{cases} \Rightarrow v_z = \frac{E_0}{B_0}\sin(\omega_c t)$

The Larmor motion is the same as before, but there is superimposed a drift of the guiding center in the y direction. The particle moves in the y-z plane. This is the so-called **E**x**B** drift.

3.

(a) The emf: $\varepsilon = -\frac{d\phi}{dt} = -Bl\frac{dx}{dt} = -Blv = IR \Rightarrow I = -\frac{Blv}{R}$ (to compensate the change of flux)

The current flow is counter-clockwise.

(b)

$\mathbf{F} = q\mathbf{v}\times\mathbf{B} = I\mathbf{l}\times\mathbf{B} = -\frac{B^2l^2v}{R} \leftarrow$ (to the left, to slow down the flux change)

(c)

$F = m\frac{dv}{dt} = -\frac{B^2l^2}{R}v \;\Rightarrow\; v = v_0 e^{-\frac{B^2l^2}{mR}t}$

4.

(a) $\begin{matrix} \nabla\cdot\mathbf{D} = \rho_{\mathrm{f}} & \nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0 \\ \nabla\cdot\mathbf{B}=0 & \nabla\times\mathbf{H}-\frac{\partial\mathbf{D}}{\partial t}=\mathbf{J}_{\mathrm{f}} \end{matrix}$,

where $\mathbf{D} = \varepsilon_0\mathbf{E}+\mathbf{P} = \varepsilon_0(1+\chi_e)\mathbf{E} = \varepsilon\mathbf{E}$ and $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}-\mathbf{M} \quad\Rightarrow\quad \mathbf{B} = \mu_0(1+\chi_{\mathrm{m}})\mathbf{H} = \mu\mathbf{H}$

(b) For linear media, $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$

$$\begin{matrix} D_1^\perp - D_2^\perp = 0 & \mathbf{E}_1^{//}-\mathbf{E}_2^{//}=0 \\ B_1^\perp - B_2^\perp = 0 & \mathbf{H}_1^{//}-\mathbf{H}_2^{//}=0 \end{matrix} \;\rightarrow\; \begin{matrix} \varepsilon_1E_1^\perp - \varepsilon_2E_2^\perp = 0 & \mathbf{E}_1^{//}-\mathbf{E}_2^{//}=0 \\ B_1^\perp - B_2^\perp = 0 & \frac{1}{\mu_1}\mathbf{B}_1^{//}-\frac{1}{\mu_2}\mathbf{B}_2^{//}=0 \end{matrix}$$

(c) Conservation of charge

$\frac{\partial\rho}{\partial t} = -\nabla\cdot\mathbf{J}$, where ρ is the charge density and $\mathbf{J}$ is the current density.

Conservation of energy

$$\frac{\partial}{\partial t}(u_{\text{mech}} + u_{\text{em}}) = -\nabla \cdot \mathbf{S},$$

where u_{mech} is the mechanical energy density, u_{em} is the electromagnetic energy density, and $\mathbf{J}$ is the Poynting vector.

Conservation of momentum

$$\frac{\partial}{\partial t}(\mathbf{g}_{\text{mech}} + \mathbf{g}_{\text{em}}) = -\nabla \cdot (-\overleftrightarrow{\mathbf{T}}),$$

where $\mathbf{g}_{\text{mech}}$ is the mechanical momentum density, $\mathbf{g}_{\text{em}}$ is the electromagnetic momentum flux density, and $\overleftrightarrow{\mathbf{T}}$ is the Maxwell stress tensor.

5.

(a) The boundary condition:

$$B_{in}^{\perp} - B_{out}^{\perp} = 0, \quad \frac{1}{\mu_{in}}\mathbf{B}_{in}^{//} - \frac{1}{\mu_{out}}\mathbf{B}_{out}^{//} = 0, \quad \text{where } \mathbf{B}_{in}^{//} = 0 \text{ and } B_{in}^{\perp} = \mu_0 M$$

So $\mathbf{B}_{out}^{//} = 0$ and $B_{out}^{\perp} = \mu_0 M$ in z direction .

(b)

$$T_{ij} \equiv \varepsilon_0 (E_i E_j - \frac{1}{2}\delta_{ij}E^2) + \frac{1}{\mu_0}(B_i B_j - \frac{1}{2}\delta_{ij}B^2)$$

$E_x = E_y = E_z = 0$ and $B_x = B_y = 0$

$B_z = \mu_0 M$ at the bottom surface

$B_z = 0$ at the top surface

$$T_{xx} = T_{yy} = -T_{zz} = -\frac{1}{2\mu_0}B_z^2, \quad \text{other } T_{ij} = 0$$

$$\mathbf{F} = \oint_S \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} - \varepsilon_0 \mu_0 \frac{d}{dt}\int_V \mathbf{S} d\tau = \oint_S \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}, \text{ since } \mathbf{S} = 0.$$

From the symmetric consideration, only z component is nonvanished.

Consider the upper half:

$$F_y = \int_{\text{top}} T_{zz} d\mathbf{a} + \int_{\text{bottom}} T_{zz} d\mathbf{a} + \int_{\text{cylinder}} T_{zz} d\mathbf{a} = \int_{\text{bottom}} T_{zz} d\mathbf{a} = \frac{\mu_0 M^2}{2}\pi a^2 \text{ (attractive force)}$$

(c)

Consider a very small gap.

$$U = U_0 + \frac{1}{2\mu_0}(\mu_0 M)^2(\pi a^2 \cdot \Delta z) = U_0 + \frac{\pi a^2 \mu_0}{2} M^2 \Delta z$$

$$\mathbf{F} = -\nabla U = -\frac{\pi a^2 \mu_0}{2} M^2 \hat{\mathbf{z}}, \text{ same as (b). (attractive force)}$$